Esercizio 10

Calcolare il flusso del rotore del campo

$$\overrightarrow{F}(x,y,z) = xy\,\overrightarrow{i}_1 + xy\,\overrightarrow{i}_2 + 0\,\overrightarrow{i}_3$$

attraverso la regione piana $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$, con

$$\mathcal{S}_1 = \left\{(x,y) \in \mathbb{R}^2 \,:\, 0 \le x \le 1,\ 0 \le y \le \sin\left(\frac{\pi}{2}x\right)\right\},$$

$$\mathcal{S}_2 = \left\{(x,y) \in \mathbb{R}^2 \,:\, 1 \le x \le \sqrt{2},\ 0 \le y \le \sqrt{2 - x^2}\right\}.$$

Applichiamo la formula di Stokes, osservando che il bordo $\Gamma = \partial\mathcal{S}$ è

$$\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$$

con

$$\begin{cases} \Gamma_1 \text{ segmento fra } (0,0) \text{ e } (\sqrt{2},0) \\ \Gamma_2 \text{ arco circonf. } x^2 + y^2 = 2 \text{ da } (\sqrt{2},0) \text{ a } (1,1) \\ \Gamma_3 \text{ curva } y = \sin\left(\frac{\pi}{2}x\right) \text{ da } (1,1) \text{ a } (0,0) \end{cases}$$

Quindi

$$\int_{\mathcal{S}} \overrightarrow{\mathrm{rot}}(\overrightarrow{F}) \cdot \overrightarrow{n}\, \mathrm{d}S$$
$$= \oint_{\Gamma} \overrightarrow{F} \cdot \mathrm{d}\Gamma$$
$$= \int_{\Gamma_1} \overrightarrow{F} \cdot \mathrm{d}\Gamma_1 + \int_{\Gamma_2} \overrightarrow{F} \cdot \mathrm{d}\Gamma_2 + \int_{\Gamma_3} \overrightarrow{F} \cdot \mathrm{d}\Gamma_3$$

- Parametrizzo Γ_1 con

$$\overrightarrow{r}_1(t) = t\,\overrightarrow{i}_1 + 0\,\overrightarrow{i}_2, \quad t \in [0, \sqrt{2}]$$

quindi $\overrightarrow{F}(\overrightarrow{r}_1(t)) \equiv 0$ su $[0, \sqrt{2}]$, da cui

$$\int_{\Gamma_1} \overrightarrow{F} \cdot \mathrm{d}\Gamma_1 = 0$$

- Parametrizzo Γ_2 con

$$\overrightarrow{r}_2(t) = \sqrt{2}\cos(t)\,\overrightarrow{i}_1 + \sqrt{2}\sin(t)\,\overrightarrow{i}_2, \quad t \in \left[0, \frac{\pi}{4}\right]$$

quindi

$$\begin{cases} \overrightarrow{r}'_2(t) = -\sqrt{2}\sin(t)\,\overrightarrow{i}_1 + \sqrt{2}\cos(t)\,\overrightarrow{i}_2 \\ \\ \overrightarrow{F}(\overrightarrow{r}_2(t)) = 2\cos(t)\sin(t)\,\overrightarrow{i}_1 + 2\cos(t)\sin(t)\,\overrightarrow{i}_2 \end{cases}$$

da cui

$$\int_{\Gamma_2} \overrightarrow{F} \cdot \mathrm{d}\Gamma_2$$

$$= 2\sqrt{2} \int_0^{\pi/4} \left(-\cos(t) \sin^2(t) + \cos^2(t) \sin(t) \right) \mathrm{d}t$$

$$= 2\sqrt{2} \left[-\frac{\sin^3(t)}{3} - \frac{\cos^3(t)}{3} \right]_0^{\pi/4}$$

$$= \frac{(2-\sqrt{2})\sqrt{2}}{3}$$

- Osservo che

$$\int_{\Gamma_3} \overrightarrow{F} \cdot \mathrm{d}\Gamma_3 = -\int_{\tilde{\Gamma}_3} \overrightarrow{F} \cdot \mathrm{d}\tilde{\Gamma}_3$$

con

$$\tilde{\Gamma}_3 : \ \overrightarrow{r}_3(t) = t \, \overrightarrow{i}_1 + \sin\left(\frac{\pi}{2} t\right) \overrightarrow{i}_2, \quad t \in [0, 1]$$

quindi

$$\begin{cases} \overrightarrow{r}'_3(t) = \overrightarrow{i}_1 + \frac{\pi}{2}\cos\left(\frac{\pi}{2}t\right)\overrightarrow{i}_2 \\ \\ \overrightarrow{F}(\overrightarrow{r}_3(t)) \cdot \overrightarrow{r}'_3(t) \\ \qquad = t\sin\left(\dfrac{\pi}{2}t\right)\overrightarrow{i}_1 + \dfrac{\pi}{2}t\sin\left(\dfrac{\pi}{2}t\right)\cos\left(\dfrac{\pi}{2}t\right)\overrightarrow{i}_2 \end{cases}$$

da cui, ponendo $z = \frac{\pi}{2}t$

$$\int_{\tilde{\Gamma}_3} \overrightarrow{F} \cdot \mathrm{d}\tilde{\Gamma}_3$$

$$= \int_0^{\pi/2} \frac{2}{\pi}\left(\frac{2}{\pi}z\sin(z) + z\sin(z)\cos(z)\right)\mathrm{d}z$$

$$= \ldots = \frac{4}{\pi^2} + \frac{1}{4}$$

- Allora

$$\int_{\mathcal{S}} \overrightarrow{\mathrm{rot}}(\overrightarrow{F}) \cdot \overrightarrow{n}\,\mathrm{d}S = \frac{2\sqrt{2}}{3} - \frac{4}{\pi^2} - \frac{11}{12}.$$